Section 7.8:
Improper Integrals

Type I: Infinite Integrals

Story behind $\int_1^\infty \frac{1}{x^2}\, dx$ …

Ex 1: Find $\int_1^\infty \frac{1}{x^2}\, dx$

Type I: Infinite Integrals

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

Type I: Infinite Integrals

1 Definition of an Improper Integral of Type 1

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

Type I: Infinite Integrals

Def:

1. $\int_a^\infty f(x)\,dx$ is convergent it is a (finite) number. I.e. the limit that defines it exists. Otherwise it's divergent.

2. $\int_{-\infty}^b f(x)\,dx$ is convergent it is a (finite) number. I.e. the limit that defines it exists. Otherwise it's divergent.

1 Definition of an Improper Integral of Type 1

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

Type I: Infinite Integrals

1 Definition of an Improper Integral of Type 1

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

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In part (c) any real number a can be used (see Exercise 76).

Def:

3. If both $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ are both convergent for some number a, then $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent. Otherwise $\int_{-\infty}^{\infty} f(x)\,dx$ is divergent.

Type I: Infinite Integrals

Ex 2: Find $\int_1^\infty \frac{1}{x}\,dx$ (compare to ex 1)

Type I: Infinite Integrals

Ex 3: Find $\int_{-\infty}^{0} xe^{x}\,dx$

Type I: Infinite Integrals

Ex 4: Find $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

Type I: Infinite Integrals

Ex 5: Find $\int_{-\infty}^{\infty} \sin(x)\, dx$

Type I: Infinite Integrals

Ex 6: For what values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

2 $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Type II: Discontinuous Integrands

Story behind $\int_{-2}^{5} \frac{1}{x}\,dx$ …

Type II: Discontinuous Integrands

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

Type II: Discontinuous Integrands

3 Definition of an Improper Integral of Type 2

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

Type II: Discontinuous Integrands

<u>Def</u>:

4. If f is continuous on $[a, b)$ but not continuous at b, then $\int_a^b f(x)\,dx$ is <u>convergent</u> it is a (finite) number. I.e. the limit that defines it exists. Otherwise it's <u>divergent</u>.

5. If f is continuous on $(a, b]$ but not continuous at a, then $\int_a^b f(x)\,dx$ is <u>convergent</u> it is a (finite) number. I.e. the limit that defines it exists. Otherwise it's <u>divergent</u>.

Type II: Discontinuous Integrands

3 Definition of an Improper Integral of Type 2

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Def:

6. If f is continuous everywhere on $[a, b]$ except at a number c between a and b, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent, then $\int_a^b f(x)\,dx$ is convergent. Otherwise $\int_a^b f(x)\,dx$ is divergent.

Type II: Discontinuous Integrands

Ex 7: Find $\int_2^5 \frac{1}{\sqrt{x-2}}\, dx$

Type II: Discontinuous Integrands

Ex 8: Is $\int_0^{\pi/2} \sec(x)\, dx$ convergent or divergent?

Type II: Discontinuous Integrands

Ex 9: Find $\int_0^3 \frac{1}{x-1}\, dx$

Type II: Discontinuous Integrands

Ex 10: Find $\int_0^1 \ln(x)\,dx$

Comparison Test For Improper Integrals

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Comparison Test For Improper Integrals

Ex 11: Is $\int_0^\infty e^{-x^2} dx$ convergent or divergent?

Comparison Test For Improper Integrals

Ex 12: Is $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ convergent or divergent?